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We have seen two classes of states, both forming a complete set, namely those which are created by acting with local operators on the vacuum, $\mathcal{O}^{(j_-,j_+)}_{\alpha_1\ldots\alpha_{2j_-}\dot\alpha_1\ldots\dot\alpha_{2j_+}}(x)|0\rangle$, and those which are associated to Poincaré irreps such as $|p\,\sigma\rangle_{mj}$ and $|p\,\lambda\rangle$ (massive & massless respectively). (sometime we will distinguish them by σ v.s. λ: $|p\,\sigma\rangle$ massive, $|p\,\lambda\rangle$ massless)

Let's understand now their relation by studying the overlap

$$(1)\qquad \boxed{\langle 0|\,\mathcal{O}^{(j_-,j_+)}_{A}(x)\,|p\,\sigma\rangle_{mj} = \,?}$$

Casimir SL(2,C)

label state within irrep, 2j+1 values

casimir of the irrep

collective index, e.g. $\alpha_1\ldots\alpha_{2j_-}\dot\alpha_1\ldots\dot\alpha_{2j_+}$ or tensorial $\mu,\nu\ldots$, taking $(2j_-+1)(2j_++1)$ values

"wave-function" overlap or "Polarizations"

(connect Lorentz to LG indices)

which tells us how much of $|p\,\sigma\rangle_{mj}$ is found in $\mathcal{O}^\dagger(x)|0\rangle$. Likewise, we can ask how much $|p\,\lambda\rangle_{m=0}$ is found in $\mathcal{O}^\dagger(x)|0\rangle$. Let's focus on the massive states first.

Remark:

The x^μ-dependence is easy to extract thanks to translation invariance

$$(2)\qquad \mathcal{O}^{(j_-,j_+)}_A(x) = e^{iP\cdot x}\,\mathcal{O}^{(j_-,j_+)}_A(0)\,e^{-iP\cdot x} \qquad e^{-iP\cdot x}|k\rangle = e^{-ik\cdot x}|k\rangle$$

either $|k\,\sigma\rangle_{mj}$ or $|k\,\lambda\rangle_{m=0}$

$$(3)\qquad \boxed{\langle 0|\,\mathcal{O}^{(j_-,j_+)}_A(x)\,|k\,\sigma\rangle_{mj} = e^{-ik\cdot x}\,\langle 0|\,\mathcal{O}^{(j_-,j_+)}_A(0)\,|k,\sigma\rangle_{mj}}$$

The (3) means that all these wavefunction overlap satisfy Klein-Gordon eq.

(4) $$\boxed{(\Box + m^2)\langle 0|\,\vartheta_A^{(j_-,j_+)}(x)\,|\kappa\,\sigma\rangle_{m\,j} = 0}$$ Klein-Gordon eq.

Comment: repeating the argument for the overlaps of massless multiplets $|\kappa\,\lambda\rangle_{m=0}$ one finds $\langle 0|\,\vartheta(x)\,|\kappa\,\lambda\rangle_{m=0} = e^{-i\kappa x}\langle 0|\,\vartheta(0)\,|\kappa\,\lambda\rangle_{m=0} \Rightarrow \Box\,\langle 0|\vartheta|\kappa\lambda\rangle_{m=0} = 0$

We can thus focus on $\langle 0|\,\vartheta_A^{(j_-,j_+)}(0)\,|\kappa\,\sigma\rangle_{m\,j}$ (or $\langle 0|\,\vartheta_A^{(j_-,j_+)}(0)\,|\kappa\,\lambda\rangle_{m=0}$).

In fact, we can simplify it further by recalling that

(5) $$|\kappa\,\sigma\rangle_{m\,j} \equiv U(L(\kappa,\bar{\kappa}))\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$$ $\left(|\kappa\,\lambda\rangle_{m=0} = U(L(\kappa,\bar{\kappa}))\,|\bar{\kappa}\,\lambda\rangle_{m=0}\right)$

$$\Downarrow$$

(6) $$\langle 0|\,\vartheta_A^{(j_-,j_+)}(0)\,U(L(\kappa,\bar{\kappa}))\,|\bar{\kappa}\,\sigma\rangle_{m\,j} = \underbrace{\langle 0|\,U(L)}_{\langle 0|}\,\underbrace{U^{-1}(L)\,\vartheta_A^{(j_-,j_+)}(0)\,U(L)}\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$$

(7) $$\boxed{U^{-1}(\Lambda)\,\vartheta_A^{(j_-,j_+)}(0)\,U(\Lambda) = D_A^{(j_-,j_+)\,B}(\Lambda)\,\vartheta_B^{(j_-,j_+)}(0)}$$ provide linear irrep

$$\Downarrow$$

(8) $$\boxed{\langle 0|\,\vartheta_A^{(j_-,j_+)}(0)\,U(L(\kappa,\bar{\kappa}))\,|\bar{\kappa}\,\sigma\rangle_{m\,j} = D^{(j_-,j_+)\,B}_{\;A}(L(\kappa,\bar{\kappa}))\,\langle 0|\,\vartheta_B^{(j_-,j_+)}(0)\,|\bar{\kappa}\,\sigma\rangle_{m\,j}}$$

Remark: The (8) is saying that wave-function overlap is nothing but the boosted + rotated wave-function overlap $\langle 0|\,\vartheta_B^{(j_-,j_+)}(0)\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$.
That is, it's enough to determine $\langle 0|\,\vartheta_B^{(j_-,j_+)}(0)\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$ in the preferred frame $|\bar{\kappa}\,\sigma\rangle_{m\,j}$ (at the preferred point $x^\mu=0$) because we can get $\langle 0|\,\vartheta_B^{(j_-,j_+)}(0)\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$ by covariantly boosting it and rotating it with $D(L)$ (and we can further get $\langle 0|\,\vartheta(x)\,|\kappa\,\sigma\rangle$ by translating it).

Let's determine then $\langle 0|\,\vartheta_B^{(j_-,j_+)}(0)\,|\bar{\kappa}\,\sigma\rangle_{m\,j}$ by pure geometric (group theory) reasoning

- $|\bar{\kappa}\,\sigma\rangle_{m\,j}$ spin-j irrep of $SU(2)_{L+R}$ (rotations) $\subset$ $SL(2,\mathbb{C})$ (Lorentz) $\sim SU(2)_L \otimes SU(2)_R$

- $\vartheta_A^{(j_-,j_+)}(0)$ irrep of $SL(2,\mathbb{C})$ $\Longrightarrow$ reducible with respect subgroup $SU(2)_{L+R}$

(9) $\boxed{\vec{J} = \vec{J}_+ + \vec{J}_-}$ $\qquad \vartheta_A^{(j_-,j_+)} = \bigoplus_{|j_+-j_-|\leq J \leq j_++j_-} \vartheta_{J_3}^{(J)}$ $\qquad J_3 = -J, \dots, +J$

are more explicitly

irreps w.r.t. $SU(2)_{L+R}$

(10) $$\boxed{\vartheta_A^{(j_-,j_+)}(0) = \sum_{J,\,J_3} C_A^{(j_-,j_+)\,J\,J_3}\,\vartheta_{J_3}^{(J)}(x)}$$

Clebsch-Gordan coefficients

(that we will see below are invariant tensors by matching the transf. on r.h.s & l.h.s)

Example:

(11) $A_\mu(0) = (1/2, 1/2) = 0 \oplus 1$

scalar wrt rotation; 3D vector wrt rotations

$A_\mu(0) = A_0\,\delta_\mu^0 + A_i\,\delta_\mu^i \Longrightarrow \begin{cases} C_\mu^{(1/2,1/2)\,0\,0} = \delta_\mu^0 \\ C_\mu^{(1/2,1/2)\,1\,i} = \delta_\mu^i \end{cases}$ (linear basis vs helicity basis)

From (10) we can extract $\langle 0|\vartheta_A^{(j_-,j_+)}(0)|\bar{\kappa}\,\sigma\rangle_{m\,j}$ in terms of Clebsch-Gordan coef.

(12) $$\langle 0|\vartheta_A^{(j_-,j_+)}(0)|\bar{\kappa}\,\sigma\rangle_{m\,j} \overset{(10)}{=} \sum_{J\,J_3} C_A^{(j_-,j_+)\,J\,J_3}\,\langle 0|\underbrace{\vartheta_{J_3}^{(J)}(0)}_{J\text{- irrep of } SU(2)_{L+R}}|\underbrace{\bar{\kappa}\,\sigma\rangle_j}_{j\text{-irrep of } SU(2)_{L+R}}$$

$\Downarrow$ $\delta_j^J\,\delta_{J_3}^{\sigma}\,z_j$ by Wigner-Eckart theorem (or Schur's lemma)

$\Downarrow$

(13) $$\boxed{\langle 0|\vartheta_A^{(j_-,j_+)}(0)|\bar{\kappa}\,\sigma\rangle_{m\,j} = C_A^{(j_-,j_+)\,j\,\sigma}\,z_j(\kappa^2)}$$

[[it should really be $\propto \delta_{-J_3}^{\sigma}$ but we are summing over J_3 so not really important, just redefine $C^{-J_3} \to C^{J_3}$]]

($z_j = z_j(\bar{\kappa})$ but $\bar{\kappa}$ the same for all orbit $\kappa^2 = \bar{\kappa}^2$ so we can write $z_j(\kappa^2)$)

Lesson:

The wave function overlap at $x^\mu=0$ and $k^\mu=\bar{k}^\mu$ is fixed in terms of group-theory object — the Clebsch-Gordan $C_A^{(j_-,j_+)j\sigma}$ — up to an overall normalization $z_j(k^2)$

"Kinematics" $\longrightarrow$ fix tensor structure

"dynamics" $\longrightarrow$ fix the $z_j(k^2)$

Since Kinematics fixes almost everything in terms of known objects at $x^\mu=0$ & $k^\mu=\bar{k}^\mu$, and since going at $x^\mu\neq 0$ and generic k^μ is obtained by covariant transformations eq.(8) and (3)

$$(14) \qquad \langle 0|\, \mathcal{O}_A^{(j_-,j_+)}(x)\,|k\,\sigma\rangle_{m_j} = e^{-ikx} D^{(j_-,j_+)B}{}_{A}(L(k,\bar{k}))\, C_B^{(j_-,j_+)j\sigma}\, z_j(k^2)$$

Known, same in any theory (e.g. free theory)

vary depending on the theory: dynamical

Since all those factors in (14) are known, we will see that we can express the $\langle 0|\, \mathcal{O}_A^{(j_-j_+)}(x)\,|k\,\sigma\rangle_{m_j}$ as solution of geometric, group-theory, differential constraints.

To this end, it's convenient to make manifest some properties of Clebsch-Gordan

– Clebsch-Gordan Key properties $\boxed{1}$ and $\boxed{2}$

$\boxed{1}$ $\quad C_A^{(j_-,j_+)JJ_3}$ are invariant tensors w.r.t. rotations

$$(14) \qquad \boxed{C_A^{(j_-,j_+)j\sigma} = D^{(j_-,j_+)B}{}_{A}(R^{-1})\, C_B^{(j_-,j_+)j\sigma'}\, D^{(j)}{}_{\sigma'}{}^{\sigma}(R)}$$

transform each index appropriately gives back C's.

$$\Updownarrow$$

$$(15) \qquad D^{(j_-,j_+)}{}_{A}{}^{B}(R)\, C_B^{(j_-,j_+)j\sigma} = C_A^{(j_-,j_+)j\sigma'}\, D^{(j)}{}_{\sigma'}{}^{\sigma}(R)$$

$\left(D^{(j_-,j_+)}_{(R)}\cdot C = C\, D^{(j)}_{(R)}\right.$ in index free notation$\left.\right)$

[C's are as close to the identity as possible, in suitable basis, since $D^{(j_-,j_+)}$ are not $SU(2)_{L+R}$-irreps]

<u>Example</u> $A_\mu = (1/2, 1/2) = 0 \oplus 1 = \delta_\mu^{\ 0} A_0 + \delta_\mu^{\ i} A_i$

(16)
$$C_\mu^{(1/2,1/2)\,1\,i} = \delta_\mu^{\ i} = \begin{pmatrix} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0\,0\,0 \\ \hline \mathbb{1}_{3\times3} \end{pmatrix} \qquad \delta_\mu^{\ 0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = C_\mu^{(1/2,1/2)\,0\,0}$$

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^{-1}_{3\times3} \end{array}\right) \cdot \begin{pmatrix} 0\ 0\ 0 \\ \hline \mathbb{1}_{3\times3} \end{pmatrix} \cdot R_{3\times3} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R^{-1}_{3\times3} \end{array}\right) \begin{pmatrix} 0\ 0\ 0 \\ \hline R_{3\times3} \end{pmatrix} = \begin{pmatrix} 0\ 0\ 0 \\ \hline \mathbb{1}_{3\times3} \end{pmatrix}$$

Proof of $\boxed{1}$: insert $U^{-1}(R)\,U(R) = \mathbb{1}$ in (13), namely

$$\cancel{z_j}\, C_A^{(j_-,j_+)\,j\,\sigma} = \langle 0|\, \mathcal{O}_A^{(j_-,j_+)}(0)\, |\bar{k}\,\sigma\rangle_{jm} = \langle 0|\, \mathcal{O}_A^{(j_-,j_+)}(0)\, U(R^{-1})\, U(R)\, |\bar{k}\,\sigma\rangle_{jm}$$
$$= D^{(j_-,j_+)}{}_A{}^B(R^{-1})\, D^{(j)}{}_{\sigma'}{}^{\sigma}(R)\, \langle 0|\, \mathcal{O}_B^{(j_-,j_+)}(0)\, |\bar{k}\,\sigma'\rangle_{jm} = D^{(j_-,j_+)}{}_A{}^B(R^{-1})\, C_B^{(j_-,j_+)\,j\,\sigma'}\, D^{(j)}{}_{\sigma'}{}^{\sigma}(R)\, \cancel{z_j}$$

$\boxed{2}$ $\quad C_A^{(j_-,j_+)\,j\,\sigma}$ are "pseudo-unitary" (not unitary, not even invertible!)

$$\boxed{\begin{aligned} &(17\text{-}a) \qquad \sum_{A\bar{A}} C^{*\,JJ_3}_{\bar{A}}\, \mathbb{P}^{\bar{A}A}\, C_A^{\ j\sigma} = \delta_j^J\, \delta_\sigma^{J_3}\, p^{(j)} \\ &(17\text{-}b) \qquad \sum_{j\sigma} C_A^{\ j\sigma}\, p^{(j)}\, C^{*\,j\sigma}_{\bar{B}} = \mathbb{P}_{A\bar{B}} \end{aligned}}$$

$\underline{p^{(j)\,2} = 1}$

where we are using a shorthand notation where (j_-, j_+) and (j_+, j_-) are suppressed

(18)
$$\left(C_A^{(j_-,j_+)\,JJ_3}\right)^* \equiv C^{*\,JJ_3}_{\bar{A}} \qquad (\in (j_+, j_-))$$

and where $\mathbb{P}_{A\bar{B}}$ and $\mathbb{P}^{\bar{A}B}$ are the irrep for parity $\mathcal{P}: (j_-, j_+) \longleftrightarrow (j_+, j_-)$

(19)
$$\begin{cases} \mathcal{O}_A^{(j_-,j_+)}(0) \xrightarrow{\ \mathcal{P}\ } \mathbb{P}_{A\bar{B}}\, \mathcal{O}^{\bar{B}\,(j_+,j_-)}(0) \\ \mathcal{O}^{\bar{A}\,(j_+,j_-)}(0) \longrightarrow \mathbb{P}^{\bar{A}B}\, \mathcal{O}_B^{(j_-,j_+)}(0) \end{cases} \qquad \mathbb{P}_{A\bar{B}}\, \mathbb{P}^{\bar{B}C} = \delta_A^C \qquad \mathbb{P}^{\bar{A}B}\, \mathbb{P}_{B\bar{C}} = \delta_{\bar{C}}^{\bar{A}}$$

($\nwarrow$ abuse of notation, e.g. A index in $\mathbb{P}_{A\bar{B}}$ belongs to (j_+, j_-) like $\bar{B}$) $\quad (\mathcal{P}^2 = 1)$

e.g. $\begin{cases} \psi_\alpha = (\tfrac12, 0) \xrightarrow{P} \chi^{\dot\alpha} = \chi^{\dot\beta} \delta^{\dot\alpha}_{\dot\beta} = (0, \tfrac12) & \chi^{\dot\alpha} = (0, \tfrac12) \xrightarrow{P} \psi_\alpha = \delta_\alpha{}^\beta \psi_\beta = (\tfrac12, 0) \\ \chi_{\dot\alpha} \xrightarrow{P} \psi^\alpha = \delta^\alpha_\beta \psi^\beta & \psi^\alpha \xrightarrow{P} \chi_{\dot\alpha} = \delta_{\dot\alpha}{}^{\dot\beta} \chi_{\dot\beta} \end{cases}$

which we can, with abuse of notation, express as $\begin{cases} \psi_\alpha \to \sigma^0_{\alpha\dot\beta} \chi^{\dot\beta} & \chi^{\dot\alpha} \to \bar\sigma^{0\dot\alpha\beta} \psi_\beta \\ \psi^\alpha \to \chi_{\dot\beta} \bar\sigma^{0\dot\beta\alpha} & \chi_{\dot\alpha} \to \psi^\beta \sigma^0_{\beta\dot\alpha} \end{cases}$

Another example is the $(\tfrac12, \tfrac12)$-irrep,

$A_\mu = (\tfrac12, \tfrac12) \xrightarrow{P} P_\mu{}^\nu A_\nu \qquad P_\mu{}^\nu = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_\mu{}^\nu = \eta^{\mu\nu}$ which in spinorial indices give

$A_{\alpha\dot\alpha} \equiv \sigma^\mu_{\alpha\dot\alpha} A_\mu \xrightarrow{P} \sigma^\mu_{\alpha\dot\alpha} P_\mu{}^\nu A_\nu = \sigma^0_{\alpha\dot\beta} \bar\sigma^{\nu\dot\beta\gamma} \sigma^0_{\gamma\dot\alpha} A_\nu = \sigma^0_{\alpha\dot\beta} \varepsilon^{\gamma\beta} \varepsilon^{\dot\beta\dot\delta} \sigma^0_{\delta\dot\alpha} A_{\beta\dot\delta}$

$\sigma^0 \bar\sigma^\nu \sigma^0 = \sigma^\mu P_\mu{}^\nu$ (really)

(again abuse of notation)

$\overset{P}{\to} (\underbrace{\sigma^0_{\alpha\dot\beta}\, \sigma^0_{\gamma\dot\alpha}}_{P_{A\bar B}}) A^{\gamma\dot\beta}$ (as expected from tensor product)

There could also be phases η_P with $\eta_P^2 = 1$, so that $P_{A\bar B}$ & $P^{\bar A B}$ are strings of σ^0 and $\bar\sigma^0$ and possible phases, in the α's, $\dot\alpha$'s basis whereas on tensor act with $P_\mu{}^\nu$.

pseudo-unitarity of C's (equivalent to (17))

$$\boxed{\begin{aligned} &C^{*Aj\sigma} \overset{\text{definit.}}{\equiv} \sum_{\bar A} C^{*}_{\bar A}{}^{j\sigma} P^{\bar A A} p^{(j)} \overset{P^2=1}{\Longleftrightarrow} \sum_A C^{*Aj\sigma} P_{A\bar A}\, p^{(j)} = C^{*}_{\bar A}{}^{j\sigma} \\ &\sum_A C^{*AJJ_3} C_A{}^{j\sigma} = \delta^J_j \delta^{J_3}_\sigma \qquad \sum_{j\sigma} C_A{}^{j\sigma} C^{*Bj\sigma} = \delta_A^B \end{aligned}} \quad (20)$$

Comments.

- Actual unitary conditions would be $U^\dagger U = \mathbb{1} = U U^\dagger$ with U invertible. This can't be fullfilled by C's because (i) not invertible, and (ii) the complex conjugation send $(j_-, j_+) \xrightarrow{*} (j_+, j_-)$
- The $\mathcal{D}^{(j_-,j_+)} \oplus \mathcal{D}^{(j_+,j_-)}$ provide irrep $\mathbb{P}$ for parity: this induces irreps labelled by $p^{(j)}$ with respect the rotation subgroup. Since $[P, \vec J] = 0$ the $p^{(j)}$ are just numbers

Example: $A_\mu = (1/2, 1/2) = 0 \oplus 1 = \delta^0_\mu A_0 + \delta^i_\mu A_i$

$\delta^0_\mu = C_\mu^{(1/2,1/2)00}$, $\delta^i_\mu = C_\mu^{(1/2,1/2)1i}$

(21) (17-a) analog for vector basis

$$\begin{cases} J=j=1 & \delta^i_\mu \eta^{\mu\nu} \delta^j_\nu = \eta^{ij} = -\delta^{ij} \Rightarrow P^{(j=1)} = -1 \\ J=1\ \ j=0 & \delta^i_\mu \eta^{\mu\nu}\delta^0_\nu = \eta^{i0} = 0 \\ J=0=j & \delta^0_\mu \eta^{\mu\nu}\delta^0_\nu = \eta^{00} = 1 \rightarrow P^{(j=0)} = 1 \end{cases}$$

$$P_\mu{}^\nu = \left(\begin{array}{c|c} +1 & \\ \hline & -1 \end{array}\right)$$

(17-b) analog $\left[\ \sum_i \delta^i_\mu \delta^i_\nu (-1) + \delta^0_\mu \delta^0_\nu (+) = \eta_{\mu\nu}\right.$

[[We are basically guessing the right thing by self-consistency of (17-a) and (17-b)-analogs, and the obvious intuition that μ and ν Lorentz indices in $\delta^{i,0}_\mu$ should be contracted by metrics $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$. Alternative, working with $A_{\alpha\dot\alpha} = \sigma^\mu_{\alpha\dot\alpha} A_\mu$ and $A^{\beta\dot\beta} \xrightarrow{P} \bar\sigma^{0\dot\beta\alpha}\bar\sigma^{0\dot\alpha\beta} A_{\alpha\dot\alpha}$ $A_{\alpha\dot\alpha} \xrightarrow{P} \sigma^0_{\alpha\dot\beta}\sigma^0_{\beta\dot\alpha} A^{\beta\dot\beta}$ one extracts $P_\mu{}^\nu = \eta^{\mu\nu}$]]

Proof of $\boxed{2}$: P leaves $\vec J = \vec J_+ + \vec J_-$ invariant, $P^{-1} \vec J P = \vec J$

Therefore, rotations R commute with Parity P:

(22) $D^{(j_-,j_+)}(R)_A{}^B \, \mathbb{P}_{B\bar B} = \mathbb{P}_{A\bar A} D^{(j_+,j_-)}(R)^{\bar A}{}_{\bar B}$ and $D^{(j_+,j_-)}(R)^{\bar A}{}_{\bar B} \mathbb{P}^{\bar B C} = \mathbb{P}^{\bar A B} D(R)_B{}^{(j_-,j_+)}{}_C$

(namely $U^{-1}(RP)\, \vartheta_A^{(j_-,j_+)}(0)\, U(RP) = D_A^{(j_-,j_+)B}(R)\, U^{-1}(P)\, \vartheta_B^{(j_-,j_+)}(0)\, U(P) = D_A^{(j_-,j_+)B}(R)\, \mathbb{P}_{B\bar B}\, \vartheta^{\bar B(j_+,j_-)}(0)$

$\qquad \| $

$U^{-1}(PR)\, \vartheta_A^{(j_-,j_+)}\, U(PR) = \mathbb{P}_{A\bar A}\, U^{-1}(R)\, \vartheta^{\bar A (j_+,j_-)}(0)\, U(R) = \mathbb{P}_{A\bar A} D^{(j_+,j_-)\bar A}{}_{\bar B}(R)\, \vartheta^{\bar B(j_+,j_-)}(0)$)

The (22) in index-free notation says $D^{(j_-,j_+)}(R)\,\mathbb{P} = \mathbb{P}\, D^{(j_+,j_-)}(R)$ and $D^{(j_+,j_-)}(R)\,\mathbb{P} = \mathbb{P}\, D^{(j_-,j_+)}(R)$. Now, the $C_A^{(j_-,j_+)j\sigma}$ commute as well with rotations (see property $\boxed{1}$ and eq.(15)), $D^{(j_-,j_+)}(R)\, C^{(j_-,j_+)} = C^{(j_-,j_+)} D^{(j)}(R)$.

Therefore, the $\left(\sum_{A\bar A} C^{*\,JJ_3}_{\bar A}\, \mathbb{P}^{\bar A A}\, C_A{}^{j\sigma}\right) = (C^\dagger \mathbb{P} C)$ is invariant tensor w.r.t. rotations: $D^{(J)}(R)^\dagger (C^\dagger \mathbb{P} C) D^{(j)}(R) = C^\dagger D^{(j_-,j_+)}(R)^\dagger \mathbb{P}\, D^{(j_-,j_+)}(R)\, C = C^\dagger \mathbb{P} C$.

By schur's lemma it vanishes unless $J=j$ $J_3=\sigma$ and it depends only on J, that is the eq.(17-a). Similar reasoning brings to (17-b).

Let's put this to work for some good.

Again on A_μ-example first: $A_\mu = \delta^0_\mu A_0 + \delta^i_\mu A_i = 0 \oplus 1$ w.r.t. rotat.

$\Rightarrow A_\mu$ can have non trivial overlaps with $j=0$ and $j=1$ states only.

Let's explore those two:

- $\underline{j=0}$ overlap of $A^\dagger_\mu |0\rangle$

(23) $$\langle 0| A_\mu(0) | \bar{k} \rangle_{m\, j=0} \equiv V_\mu$$ overlap of $A_\mu|0\rangle$ with spin-0 ?

$\overset{\text{Eq.(11)}}{=} \delta^0_\mu\, z_{j=0}$

$\Lambda_\mu{}^\nu (L(k,\bar{k}))$

(24) $$\langle 0| A_\mu(0) | k \rangle_{m\, j=0} = L_\mu{}^\nu(k,\bar{k}) \langle 0| A_\nu(0) | \bar{k} \rangle_{m\, j=0} = L_\mu{}^\nu(k,\bar{k})\underbrace{\delta^0_\nu}_{\shortparallel\ k_\nu/m} z_{j=0} = \frac{k_\mu}{m} z_{j=0}$$

(25) $$\boxed{\langle 0| A_\mu(x) | k \rangle_{m\, j=0} = e^{-ikx} \frac{k_\mu}{m} z_{j=0} \equiv V_\mu(x)}$$

overlap vector with spin-0 state

<u>Comment</u>: this is hardly surprising since $\langle 0| \partial_\mu \phi | k \rangle_{m\, j=0} \propto k_\mu e^{-ikx}$

<u>Remark</u>: we can recast (25) as the solution of the constraint

(26) $$\boxed{\begin{aligned} \partial_{[\mu} V_{\nu]} &= 0 \\ (\Box + m^2) V_\mu &= 0 \end{aligned}}$$ (equivalently $dV = 0$ for $V = V_\mu dx^\mu$ 1-form)

constraints on $j=0$ overlap for vector

- $\underline{j=1}$ overlap of $A^\dagger_\mu |0\rangle$

(27) $$\langle 0| A_\mu(0) | \bar{k}\, \sigma \rangle_{m\, j=1} \equiv \varepsilon^\sigma_\mu(\bar{k})$$

$= z_{j=1} \cdot \delta^\sigma_\mu$

$\Rightarrow$ $$\varepsilon^\sigma_\mu(k) \equiv \langle 0| A_\mu(0) | k\, \sigma \rangle_{m\, j=1} = L_\mu{}^\nu(k,\bar{k})\, \delta^\sigma_\nu \cdot z_{j=1}$$

Since $L_\mu{}^\nu(k,\bar{k})$ is known (we fixed it) we could just calculate $\Lambda_\mu{}^\nu \delta^\circ_\nu$

But there is a nice and instructive shortcut:

(28) $$\bar{K}^\mu \delta^\sigma_\mu = 0 \iff K^\mu L_\mu{}^\nu(K,\bar{K})\, \delta^\sigma_\nu = 0 \iff \boxed{K^\mu \varepsilon^\sigma_\mu = 0}$$

(29) $$\boxed{\partial_\mu V^\mu = 0 \qquad (\Box + m^2) V_\mu = 0} \qquad V_\mu(x) \equiv \langle 0 | A_\mu(x) | K\sigma \rangle_{m\, j=1}$$

spin-1 wavefunction overlap for A_μ

Remark: The $\varepsilon^i_\mu(\bar{K}) \propto \delta^i_\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ is a particular choice of basis known as "linear-polarizations". A more common and often convenient basis is instead the "helicity-polarizations" (or "circular") where J_3 is diagonalized. Indeed, by invariance under rotations:

(30) $$\varepsilon^\sigma_\mu(\bar{K}) = \langle 0 | A_\mu | 0 \rangle \overset{R_z^{-1} R_z}{|\bar{K}\sigma\rangle_{mj}} = (R_z^{-1})_\mu{}^\nu\, \varepsilon^\sigma_\nu(\bar{K}) \exp(-i\sigma\theta) \Rightarrow (J_3)_\mu{}^\nu \varepsilon^\sigma_\nu(K) = \sigma\, \varepsilon^\sigma_\mu$$

$$(J_3)_\mu{}^\nu = \left(\begin{array}{c|cc|c} 0 & 0 & & 0 \\ \hline 0 & \begin{matrix} 0 & i \\ -i & 0 \end{matrix} & & 0 \\ \hline 0 & 0 & & 0 \end{array}\right)_\mu{}^\nu$$

whose solutions are

(31) $$\varepsilon^{\sigma=\pm}_\mu(\bar{K}) = \frac{1}{\sqrt{2}} \begin{pmatrix}0\\1\\\mp i\\0\end{pmatrix}, \quad \varepsilon^{\sigma=0}_\mu(\bar{K}) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Connected to linear $\varepsilon^i_\mu(\bar{K})$ by an $SU(2)$ rotation (the $LG_{\bar{K}}$). Thus in either case

(32) $$\sum_\sigma \varepsilon^\sigma_\mu(\bar{K}) \varepsilon^{\sigma *}_\nu(\bar{K}) = \sum_i \varepsilon^i_\mu(\bar{K}) \varepsilon^{i*}_\nu(\bar{K}) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbb{1}_{3\times 3} \end{array}\right)_{\mu\nu} = -\eta_{\mu\nu} + \frac{\bar{K}_\mu \bar{K}_\nu}{m^2} \Longrightarrow \boxed{\sum_\sigma \varepsilon^\sigma_\mu(K) \varepsilon^{\sigma *}_\nu(K) = -\eta_{\mu\nu} + \frac{K_\mu K_\nu}{m^2}}$$

polarization sums for $j=1$ in A_μ.

This is consistent with (21): $-\sum_i \delta^i_\mu \delta^i_\nu = \eta_{\mu\nu} - \delta^0_\mu \delta^0_\nu$

another non-trivial example: massive spin-2 in $h_{\mu\nu} = h_{\nu\mu}$

(33) $$h_{\mu\nu} = (1,1) + (0,0) = 2 \oplus 1 \oplus 0 \oplus 0 \quad \text{w.r.t. rotations}$$

$(1,1)$: traceless sym. part; $(0,0)$: $h_\mu{}^\mu$

5 d.o.f. $+ 3 + 1 + 1 = 10$ d.o.f. total ok

(34) $\langle 0 | h_{\mu\nu}(0) | \vec{k}\, \sigma \rangle_{m\, j=2} \overset{?}{=} \varepsilon^{\sigma}_{\mu\nu}(\vec{k})$ $\qquad h_{\mu\nu}(0) = \left(\begin{array}{c|c} h_{00} & h_{0i} \\ \hline h_{i0} & h_{ij} \end{array}\right)$

00 spin-0

$0i$ spin-1

ii-traced spin-0

$ij - \delta_{ij}$ trace spin-2

(35) $U(R^{-1}) h_{\mu\nu}(0) U(R) = \Lambda_{\mu}{}^{\bar\mu}(R) \Lambda_{\nu}{}^{\bar\nu}(R) h_{\bar\mu\bar\nu}(0) = \left(\begin{array}{c|c} h_{00} & (R\vec{h})^T \\ \hline R\vec{h} & R\,h\,R^T \end{array}\right)$

$\Lambda(R) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) \qquad R R^T = \mathbb{1}_{3\times3}$

$R\,h\,R^T \to R_i{}^j \delta_{j\ell} R_k{}^\ell = \delta_{ik}$ invariant $\to$ spin-0

$\Downarrow$

(36) $$\boxed{\varepsilon^{\sigma}_{00}(\vec{k}) = \sum_i \varepsilon^{\sigma}_{ii}(\vec{k}) = \varepsilon^{\sigma}_{0i}(\vec{k}) = 0}$$

$\bar{k}^{\mu}\bar{k}^{\nu}\varepsilon^{\sigma}_{\mu\nu}(\vec{k}) \qquad \eta^{\mu\nu}\varepsilon^{\sigma}_{\mu\nu}(\vec{k}) \qquad k^{\mu}\varepsilon^{\sigma}_{\mu\nu}(\vec{k}) = 0$

(37) $$\boxed{\begin{array}{c} \langle 0 | h_{\mu\nu}(x) | k\,\sigma \rangle_{m\, j=2} \equiv \psi_{\mu\nu}(x) \\ \partial_{\mu}\psi^{\mu\nu}(x) = 0 \qquad \psi^{\mu}_{\mu}(x) = 0 \qquad (\Box + m^2)\psi_{\mu\nu} = 0 \end{array}}$$

spin-2 wavefunction for symmetric tensor

Lesson:

The general lesson here is that wavefunctions satisfy simple covariant constraints, on top of Klein-Gordon equations, simply because they are linear combination of Clebsch-Gordan coefficients which are invariant and pseudo-unitary tensors. The equations of motion for free fields, for which the 1-particle overlaps is all there is, are completely fixed by group theory.

For the sake of completeness let's work out the general constraint equations for any spin and fields, using

— Spin Projectors —

(38) $$\pi^{(j)\ B}_{A} \equiv \sum_\sigma C_A^{\ j\sigma}\, C^{*B\, j\sigma}$$ $\left(C^{*A\,j\sigma} = \sum_{\bar A} C^{*\ j\sigma}_{\bar A}\, \mathbb{P}^{\bar A A}\, p^{(j)} \right)$

defines orthogonal projectors (no sum on j) and (j_-, j_+) understood and fixed

(39) $$\boxed{\sum_B \pi^{(j)\ B}_{A}\, \pi^{(s)\ C}_{B} = \delta^s_j\, \pi^{(j)\ C}_{A} \qquad \sum_j \pi^{(j)\ B}_{A} = \delta_A^{\ B}}$$

$$\left[\!\left[\bullet \sum_{\sigma\sigma'} C_A^{j\sigma} \underbrace{\sum_{B} C^{*B j\sigma} C_B^{s\sigma'}}_{\delta^j_s\, \delta^\sigma_{\sigma'}} C^{*C s\sigma'} = \delta^s_j \sum_\sigma C_A^{j\sigma} C^{*C j\sigma} = \delta^s_j\, \pi^{(j)\ C}_{A} \right.\right.$$

$\delta^j_s\, \delta^\sigma_{\sigma'}$ by pseudo-unit. in (20)

$$\bullet \sum_{j\sigma} C_A^{j\sigma} C^{*Bj\sigma} \overset{\text{eq.(20)}}{=} \delta_A^{\ B} \left.\right]\!\!\!\;\,\Big]$$

(40) $$\boxed{\pi^{(j)}\pi^{(s)} = \pi^{(j)}\delta^s_{(j)} \qquad \sum_j \pi^{(j)} = \mathbb{1}}$$ in matrix notation

Now, the wavefunctions $\psi_A^{j\sigma}(\bar k) \equiv \langle 0 |\, \vartheta_A^{(j_-, j_+)}(0)\, | \bar k\, \sigma \rangle_{m_j} = \sum_j C_A^{j\sigma}$ are proportional to the clebsch-Gordan so that it follows

(41) $$\sum_B \pi^{(j)\ B}_{A}\, \psi_B^{s\sigma}(\bar k) = \delta^j_s\, \psi^{s\sigma}$$ $\left(\sum_\sigma C_A^{j\sigma} \underbrace{\sum_B C^{*Bj\sigma} C_B^{s\sigma}}_{\delta^j_s\,\delta^\sigma_\sigma} z_s = (C_A^{s\sigma} z_s)\delta^j_s \right)$

$$\Downarrow$$

(42) $$\sum_B \underbrace{(\mathbb{1} - \pi^s)}_{\pi^s_\perp = \sum_{j\neq s}\pi^j}{}_A^{\ B}\, \psi_B^{s\sigma} = \boxed{\left(\pi_\perp^{\ s} \cdot \psi^{s}(\bar k)\right)_A = 0}$$ general wavefunction constraint @ $k = \bar k$

$\pi^s_\perp \cdot \pi^s = 0$ ← orthogonal projector

which we can promptly recast into a covariant form acting with $D_A^{\ B}(L(k,\bar k))$

(43) $$\Pi^{(j)\,B}_{A}(k) \equiv D_A{}^C(L(k,\bar k))\,\underset{\parallel}{\Pi^{(j)\,D}_{C}}\,D_D{}^B(L^{-1}(k,\bar k))$$

$$\left(\Pi^{(j)\,D}_{C}(\bar k)\right)$$

$$\Downarrow$$

(44) $$\boxed{\sum_B \Pi^{(s)\,B}_{\perp\,A}(k)\,\psi^{s}_{B}(k) = 0} \qquad \Pi^{(s)}(k)\cdot\psi^{(s)}_{(k)} = 0$$

which are nicely covariant constraints, since

(45) $$D(\Lambda)\,\Pi_\perp(k) = D(\Lambda)\,D(L(k,\bar k))\,\Pi_\perp(\bar k)\,D(L^{-1}(k,\bar k))$$

$$= D(L(\Lambda k,\bar k))\,\underbrace{\underbrace{D(L^{-1}(\Lambda k,\bar k)\cdot\Lambda\cdot L(k,\bar k))}_{\in LG_{\bar k}}\,\Pi_\perp(\bar k)\,D(L^{-1}(k,\bar k)\cdot\Lambda^{-1}\cdot L(\Lambda k,\bar k))}_{\Pi_\perp(\bar k)\ \text{because } LG_{\bar k} \text{ leaves } \bar k \text{ invariant and } c\text{'s are invariant tensors}}\,D(L^{-1}(\Lambda k,\bar k))\,D(\Lambda)$$

$$\Downarrow$$

$$= D(L(\Lambda k,\bar k))\,\Pi_\perp(\bar k)\,D(L^{-1}(\Lambda k,\bar k))\,D(\Lambda)$$

so that $$D(\Lambda)\,\Pi_\perp(k)\,\psi(k) = \Pi_\perp(\Lambda k)\cdot\big(D(\Lambda)\psi(k)\big) = 0$$

(46) $$D(\Lambda)\,\Pi_\perp(k) = \Pi_\perp(\Lambda k)\,.$$

In summary: we have that wavefunction envelops satisfy

(47) $$\boxed{\begin{aligned} \Pi^{(s)}_{\perp}(k)\cdot\psi(k) &= 0 \\ (k^2-m^2)\,\psi(k) &= 0 \\ D(\Lambda)\,\Pi^{(s)}_{\perp}(k) &= \Pi^{(s)}_{\perp}(\Lambda k) \end{aligned}}$$

Covariant e.o.m. follow from group theory, they are kinematical, for any spin/field.

Let's see again our favorite example:

Example: vector $A_\mu = (1/2, 1/2)$ vs spin-1, 0

(48) $$C_\mu^{j=0} = \delta_\mu^{\ 0} \longrightarrow \Pi^{(j=0)}(\bar{K})_\mu^{\ \nu} = \delta_\mu^{\ 0}\,\delta_\rho^{\ 0}\,\eta^{\rho\nu}(+) = \delta_\mu^{\ 0}\eta^{0\nu} \Rightarrow \Pi^{(j=0)}(\bar{K})_{\mu\nu} = \delta_\mu^{\ 0}\delta_\nu^{\ 0} = \frac{\bar{K}_\mu \bar{K}_\nu}{m^2}$$

(49) $$\Rightarrow \boxed{\Pi^{(j=0)}(K)_{\mu\nu} = \frac{K_\mu K_\nu}{m^2}} \Rightarrow \Pi^{(j=0)} = 1 - \Pi^{(j=1)} = \Pi_\perp^{(j=1)}$$ indeed $K_\mu V^\mu = 0$ for $j=1$

(50) $$C_\mu^{j=1,i} = \delta_\mu^{\ i} \text{ (linear basis)} \quad \Pi^{(j=1)}(\bar{K})_\mu^{\ \nu} = \sum_i \delta_\mu^{\ i}\,\delta_\rho^{\ i}\,\eta^{\rho\nu}(-) = -\delta_\mu^{\ i}\eta^{i\nu} = \delta_\mu^{\ i}\delta_i^{\ \nu}$$

indeed

(51) $$\Pi^{(j=1)\,\nu}_{\ \mu} + \Pi^{(j=0)\,\nu}_{\ \mu} = \delta_\mu^{\ 0}\eta^{0\nu} + \delta_\mu^{\ i}\delta_i^{\ \nu} = \delta_\mu^{\ \nu}$$

and one can check $\Pi^{(j=1)}\cdot\Pi^{(j=0)} = 0 \qquad \Pi^{(j=1)}\cdot\Pi^{(j=1)} = \Pi^{(j=1)} \ldots$

Moreover,

(52) $$\Pi^{(j=1)\,\nu}_{\ \mu} = \delta_\mu^{\ \nu} - \Pi^{(j=0)\,\nu}_{\ \mu} = \delta_\mu^{\ \nu} - \frac{\bar{K}_\mu \bar{K}_\nu}{m^2}$$, confirming previous findings.

$$-\sum_i \varepsilon_\mu^{\,i}\,\varepsilon_\nu^{\,i*}$$